Solving for the Analytic Piecewise Extension of Tetration and the Super-logarithm

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Abstract

An overview of previous extensions of tetration is presented. Specific conditions for differentiability and piecewise continuity are shown. This leads to a way of generating approximations of the super-logarithm. These approximations are shown to converge to a function that satisfies two basic properties of extensions of tetration.

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Introduction

Of the operators in the sequence: addition, multiplication, exponentiation, and tetration, only the first three are well-defined analytic operations. *Tetration*, also known as the hyper4 operator, power towers, and iterated exponentiation, has only been "nicely" defined for integer y in ${}^{y}x$. Although extensions of tetration to real y have been made, those extensions have not followed simply from the properties of hyper-operations. The particular form of this property as it pertains to tetration is:

Property 1. *Iterated exponential property*

$${}^{y}x = x^{({}^{y-1}x)} \quad \text{for all real } y.$$

Tetration has two inverses, the *super-root*, and the *super-logarithm*:

$$z = {}^{y}x = \mathrm{tet}_x(y) = \mathrm{twr}_y(x)$$
$$x = \mathrm{srt}_y(z) = \mathrm{twr}_y^{-1}(z)$$
$$y = \mathrm{slog}_x(z) = \mathrm{tet}_x^{-1}(z)$$

When y is fixed, the function $\mathrm{twr}_y(x)$ is called an *order y power tower of x* [14]. When x is fixed, the function $\mathrm{tet}_x(y)$ is called a *base x tetrational function of y*. In general, it is pronounced: *x tetra y*. Inverting a power tower gives the super-root, and inverting a tetrational function gives the super-logarithm, More detail about these inverses can be found in [12], [14] and [15]. The super-logarithm has an equivalent to property (1), and can be found by applying the super-logarithm to property (1) as follows:

$$y - 1 = \mathrm{slog}_x({}^{y-1}x)$$
$$y = \mathrm{slog}_x({}^{y}x) = \mathrm{slog}_x(x^{({}^{y-1}x)}) = \mathrm{slog}_x({}^{y-1}x) + 1$$

Replacing ${}^{y-1}x$ with z, we get the equivalent property for the super-logarithm:

Property 2. *Function linearization property*

$$\mathrm{slog}_x(x^z) = \mathrm{slog}_x(z) + 1$$

This property is actually a special case of Abel's equation for *linearizing* a function, which is found in [4] and [9]. Linearization is done in the hope of iterating a function continuously, or finding the *iterative roots* of a function [6]. The generalization $\phi(f(z)) = \phi(z)+1$ can be used to iterate a function continuously, as follows:

$$\begin{aligned}
&\phi(f(z)) = \phi(z) + 1 \\
&\phi(f^n(z)) = \phi(z) + n \\
&f^n(z) = \phi^{-1}(\phi(z)+n)
\end{aligned}$$

where n could be real. In the case above, $f(z) = x^z$, and $\phi(z) = \mathrm{slog}_x(z)$. This means that if we can find the linearizing function of exponentials: $\mathrm{slog}_x(z)$, then not only can we define tetration as its inverse, but we can also use it to continuously iterate exponential functions.

The next property uses the notation $\boldsymbol{C}^n$. A function $f(x)$ is $\boldsymbol{C}^n$, if it is n-times differentiable. To be analytic, a function must be infinitely differentiable, or $\boldsymbol{C}^\infty$. All the other hyper-operations below tetration (hyper4) are $\boldsymbol{C}^\infty$ for both operands, so it is natural to require this of tetration as well. When combined with either property above, this property ensures a unique extension of tetration:

Property 3. *Infinite differentiability property*

$$f(x) \text{ is } \boldsymbol{C}^\infty \equiv \boldsymbol{D}^k f(x) \text{ exists for all integer } k.$$

Up to this point, extensions of tetration have been made that satisfy property (1), but not property (3), and extensions of tetration have been made that satisfy property (3), but property (1) only for integer y. To avoid discontinuous derivatives, and to be a valid extension of tetration or the super-logarithm, these properties are necessary. It is the goal of this paper, however, to show that these properties are sufficient to find such an extension, and that the extension found will be unique.

Background

History

Tetration has fascinated mathematicians for centuries, in part because it is as fundamental as addition or multiplication in its definition, yet was beyond the realm of calculation due to its fast growth rate. Although the calculation of tetration at low values such as ${}^{5}5$ is still a daunting task, computational advances have open the doors of solving large systems of equations involved in finding non-integer values.

Many hobbyist and professional mathematicians alike have rediscovered tetration, and asked themselves the questions: what is ${}^{0.5}2$ or ${}^{e}\boldsymbol{e}$ or ${}^{\pi}\boldsymbol{e}$? Is there an answer? If so, what is the derivative $\boldsymbol{D}_y({}^{y}x)$? These questions have motivated previous extensions of tetration, just as they have motivated this one. The advances made so far in tetration have not been by motivation alone. Many mathematicians have contributed to what is known about tetration today.

The first tetration-related proof was made by Euler, when he proved the convergence of ${}^{\infty}x$, although he used a different notation. The first person to publish the notation ${}^{y}x$ was Mauer [4], and was later popularized by Rucker in [13]. The word *tetration* was coined by Goodstein from the words *tetra* and *iteration* [10]. The sequence of operations from which tetration comes is generally known as the hyper-operations, or hyper-*n* operators. Goodstein also gives names to the hyper-operations: *pentation* for hyper5, *hexation* for hyper6, and so on. The first to write about this sequence of operations was Grzegorczyk [12], hence the sequence has also been called the Grzegorczyk hierarchy. When the hyper-operation sequence is viewed as a ternary function, it is sometimes called the Ackermann function [5]. Many people have introduced notations for this sequence of operators, and beyond, including Knuth [8], Conway [2], Munafo [11], and Bowers [1].

Euler's proof of the convergence of ${}^{\infty}x$, is useful in defining many other related functions, one of which is the Lambert *W* function, also known as the product-logarithm. There is also a relationship between these two functions and the second super-root. To overview the properties:

$$
\begin{array}{ll}
{}^{\infty}x = u & \text{where } u = x^{u} \\
W(x) = u & \text{where } u\boldsymbol{e}^{u} = x \\
\operatorname{srt}_2(x) = u & \text{where } u^{u} = x
\end{array}
$$

and using only basic algebra, the following identities can be found:

$$
\begin{array}{lll}
{}^{\infty}x & = \dfrac{W(-\log(x))}{-\log(x)} & = 1/\operatorname{srt}_2(x^{-1}) \\
{}^{\infty}(\boldsymbol{e}^{-x})x & = W(x) & = x/\operatorname{srt}_2(\boldsymbol{e}^{x}) \\
1/{}^{\infty}(x^{-1}) & = \dfrac{\log(x)}{W(\log(x))} & = \operatorname{srt}_2(x)
\end{array}
$$

showing that these functions can all be expressed in terms of each other. So if it were a matter of choice, any would work, although many Computer Algebra Systems come with the product-logarithm. Aesthetically, though, ${}^{\infty}x$ seems a "nicer" choice.

Both the infinite power tower and the product logarithm have series expansions:

$$
{}^{\infty}x = \sum_{k=1}^{\infty} \frac{(k\log(x))^{k-1}}{k!}
$$

$$
W(x) = \sum_{k=0}^{\infty} \frac{(-k)^{k-1}x^{k}}{k!}
$$

and are very well-known, well-defined analytic functions.

Applications

The most immediate applications of tetration and the super-logarithm are in the representation of large numbers, and continuously iterated exponentiation. Some applications of the approximations given in this paper that may not be immediate are:

- Approximating continuous iterations of other functions
- Validating closed-form extensions of tetration or the super-logarithm
- Developing new algebraic identities using the approximations

One application of the super-logarithm could be in population modeling. In particular, a graph of the super-logarithm looks similar to a logistic population model. Instead of stopping at some point, though, the super-logarithm continues to grow.

Extensions

One of the reasons why extending ${}^{y}x$ to real y is so difficult, is that many of the laws that are instrumental in defining exponentiation (hyper3) do not hold for tetration (hyper4). For example, exponentiation satisfies $\sqrt[n]{x} = x^{1/n}$ whereas the same is not true of tetration; $\mathrm{srt}_n(x) \neq {}^{1/n}x$. This is in part because there is no multiplicative law of exponents for tetration; ${}^{b}({}^{a}x) \neq {}^{(ab)}x$. Another aspect of exponentiation that is convenient is the change-of-base formula for logarithms. There is currently no known change-of-base formula for super-logarithms, however. The absence of these properties to fall back on means that a completely different method must be used to make progress.

Piecewise Extensions

A piecewise-defined extension of tetration uses property (1) to compute the values of tetration at all intervals, given the values in an interval of length one. So any piecewise extension of tetration must obey property (1) by definition. Also, one benefit of using a piecewise extension is that in coming up with extensions, the only part that needs to be taken into account is the *critical interval*, not the entire function. Before we get to some piecewise extensions lets define their general form:

Definition 1. *General piecewise extension of tetration*

$$
\begin{aligned}
{}^{y}x &= \log_x({}^{y+1}x) && \text{if } y \le -1 \\
{}^{y}x &= t(x, y) && \text{if } -1 < y \le 0 \\
{}^{y}x &= x^{({}^{y-1}x)} && \text{if } y > 0
\end{aligned}
$$

Definition 2. *General piecewise extension of the super-logarithm*

$$
\begin{aligned}
\mathrm{slog}_x(z) &= s(x, x^z) - 1 && \text{if } z \le 0 \\
\mathrm{slog}_x(z) &= s(x, z) && \text{if } 0 < z \le 1 \\
\mathrm{slog}_x(z) &= s(x, \log_x^m(z)) + m && \text{if } ({}^{m-1}x) < z \le ({}^{m}x),\ m > 0
\end{aligned}
$$

where $t(x, y)$ is the *critical function* of tetration, and $s(x, z)$ is the *critical function* of the super-logarithm. The most common extension, found in [4] and [15], is the extension that behaves like a line within the critical functions:

Extension 1. *Linear* $t(x, y)$

$$t(x, y) = y + 1 \quad \text{if} \quad -1 < y \le 0$$

We can invert this definition to get a piecewise-defined extension of super-logarithms:

Extension 2. *Linear* $s(x, z)$

$$s(x, z) = z - 1 \quad \text{if} \quad 0 < z \le 1$$

The reason why this can be done is that the value of tetration everywhere depends only on the value of tetration where $-1 < y \le 0$, and the value of the super-logarithm everywhere depends only on the value of the super-logarithm where $0 < z \le 1$. This will help when we find a series, because the only place we need convergence is between 0 and 1.

Analytic Extensions

An analytic extension of tetration or the super-logarithm is a function that satisfies property (3). This means that these functions can be represented as an infinite series expansion. Depending on the function, it may be preferable to expand it about different points or with respect to different variables. This is a list of the expansions used here:

Definition 3. *General series extension of tetration with respect to the hyper-base*

$$^{y}x = \sum_{k=0}^{\infty} \log(x)^{k} \alpha_{k}(y)$$

Definition 4. *General series extension of tetration with respect to the hyper-exponent*

$$^{y}x = \sum_{k=0}^{\infty} \frac{y^{k}}{k!} \beta_{k}(x)$$

Definition 5. *General series extension of the super-logarithm*

$$\operatorname{slog}_x(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} v_k(x)$$

In [4], Galidakis gives an extension of the form in definition (3):

Extension 3. *Summary of Galidakis' analytic extension in* [4]

$$
\begin{array}{ll}
a_{m,n} = 1 & \text{if } m = n = 0 \\
a_{m,n} = 0 & \text{if } m = 0 \text{ and } n \neq 0 \\
a_{m,n} = \dfrac{1}{n!} & \text{if } m = 1 \\
a_{m,n} = \displaystyle\sum_{j=1}^{n} \frac{j}{n} a_{m,n-j} a_{m-1,j-1} & \text{otherwise}
\end{array}
$$

$$
\begin{array}{ll}
\phi(x) = \exp\left(\dfrac{4}{4x^2 - 1}\right) & \text{if } |x| < 1/2 \\
\phi(x) = 0 & \text{otherwise}
\end{array}
$$

$$\chi_m(x) = \frac{(a_{m,n} - a_{m-1,n})\,\phi(x - (m - 1/2))}{\displaystyle\int_{m-1}^{m} \phi(t - (m - 1/2))\,\boldsymbol{d}t}$$

$$
\begin{array}{ll}
\alpha_n(x) = 1 & \text{if } n = 0 \\
\alpha_n(x) = \displaystyle\int_0^x \sum_{m=1}^{n} \chi_m(t)\,\boldsymbol{d}t & \text{if } n \neq 0
\end{array}
$$

Extension 4. *Analytic term function related to the piecewise linear critical function*

$$
\begin{array}{ll}
\alpha_k(y) = 1 & \text{if } k = 0,\ y > 1 \\
\alpha_k(y) = \dfrac{y^k}{k!} & \text{if } 0 \leq y \leq 1 \\
\alpha_k(y) = \displaystyle\sum_{j=1}^{k} \frac{j}{k} \alpha_{k-j}(y)\,\alpha_{j-1}(y-1) & \text{otherwise}
\end{array}
$$

which uses the recurrence relation found in [4].

To show that these two analytic extensions do not satisfy property (1), we can test values. If we find one real y for which property (1) is not satisfied, then it is not satisfied for all real y. According to Galidakis [4], extension (3) converges for all x in a compact subset of the complex plane, so the series converges for $x = \boldsymbol{e}$.

Using extension (3), the values: ${}^{0.5}\boldsymbol{e} = 1.858$ and ${}^{1.5}\boldsymbol{e} = 5.613$ do not satisfy property (1), because $\boldsymbol{e}^{1.858} = 6.413 \neq 5.613$. Doing the same using extension (4), the values: ${}^{0.5}\boldsymbol{e} = 1.649$ and ${}^{1.5}\boldsymbol{e} = 5.185$ do not satisfy property (1) because $\boldsymbol{e}^{1.649} = 5.200 \neq 5.185$.

Neither of these analytic extensions satisfy property (1), so neither of these are piecewise-definable. We could use then as critical functions of a piecewise definition of tetration, but doing so would cause them to fail to be analytic (property (3)).

Is there a way to find an extension that satisfies both properties? There is, but it is not an extension of tetration itself. It is an extension of the super-logarithm, the inverse of tetration, so by finding the values of a super-logarithm that satisfies property (2), and property (3), the inverse of the super-logarithm will satisfy property (1) and property (3).

Results

Discussion

To solve the problem of extending tetration to non-integer hyper-exponents once and for all, neither property (1) nor property (3) is sufficient alone. Both must be combined in order to get a unique extension of tetration. There is already one way to ensure that property (1) be satisfied, and that is to use a piecewise extension. Now we need a way of ensuring that property (3) is satisfied. To do this we need a way of ensuring infinite differentiability. The problem is that the derivatives of some previous piecewise extensions were not continuous, so there would be a difference between the limit to a point from the left and the limit to the same point from the right. We need a way of ensuring that there is no difference between these two limits. Expressed formally:

Definition 6. *Piecewise error transform*

$$\underset{x \to c}{PWET_k}\{f(x)\} = \lim_{x \to c^+} \boldsymbol{D}^k f(x) - \lim_{x \to c^-} \boldsymbol{D}^k f(x)$$

which should be zero for all k if $f(x)$ is $\boldsymbol{C}^\infty$. To become familiar with the piecewise error transform, lets apply it to the piecewise extensions we have covered so far. First lets apply it to the piecewise extension of tetration as y approaches zero:

$$\begin{aligned}
&\underset{y \to 0}{PWET_0}\{{}^y x\} = x^{t(x,-1)} - t(x,0) = 1 - 1 = 0 \\
&\underset{y \to 0}{PWET_1}\{{}^y x\} = x^{t(x,-1)} \log(x) \boldsymbol{D}_y t(x,-1) - \boldsymbol{D}_y t(x,0) = \log(x) - 1 \\
&\underset{y \to 0}{PWET_2}\{{}^y x\} = \log(x)^2 \\
&\ldots \\
&\underset{y \to 0}{PWET_k}\{{}^y x\} = \boldsymbol{D}_y^k [x^{t(x,y-1)}]_{y=0} - \boldsymbol{D}_y^k [t(x,y)]_{y=0}
\end{aligned}$$

where the piecewise extension with linear critical function is used to evaluate tetration. The first line indicates that there is no difference between the left and right limits to zero of the function itself, whereas the second line indicates that there is a difference between the left and right limits to zero of the derivative of the function with respect to y. This could be seen in a graph as a discontinuity in the derivative, but the graph in appendix B is for $x = \boldsymbol{e}$, which makes the second line zero. This can be seen as the derivative being continuous. Although the second line can equal zero when $x = \boldsymbol{e}$, the third line will be one, and this would show itself in the graph as a discontinuity in the second derivative. From this we can determine that this extension is not $\boldsymbol{C}^{\infty}$, which makes it a non-analytic extension.

The other piecewise extension that was presented was that of the super-logarithm. Now if we apply the piecewise error transform to the piecewise extension of the super-logarithm as z approaches zero, we get: the following:

$$
\begin{aligned}
&\underset{z\to 0}{PWET_0}\{\mathrm{slog}_x(z)\} = 1 + s(x,0) - s(x,1) = 0 \\
&\underset{z\to 0}{PWET_1}\{\mathrm{slog}_x(z)\} = \boldsymbol{D}_z s(x,0) - \log(x)\boldsymbol{D}_z s(x,1) = 1 - \log(x) \\
&\underset{z\to 0}{PWET_2}\{\mathrm{slog}_x(z)\} = -\log(x)^2 \\
&\ldots \\
&\underset{z\to 0}{PWET_k}\{\mathrm{slog}_x(z)\} = \boldsymbol{D}_z^k[s(x,z)]_{z=0} - \boldsymbol{D}_z^k[s(x,x^z) - 1]_{z=0}
\end{aligned}
$$

where the piecewise extension with linear critical function is used to evaluate the super-logarithm. Again, these expressions can be seen in the graph in appendix B as a continuous red line for any x. The green line, which represents the first derivative, will be discontinuous for all x except $x = \boldsymbol{e}$. For $x = \boldsymbol{e}$, the green line is continuous, because the second expression above is zero. There is no number that makes the third expression zero, so this will be seen as a discontinuous blue line for any x.

To make an extension of tetration that satisfies property (1) and property (3), we can combine the general extensions found in definition (1) and definition (4), by using the series as the critical function $t(x,y)$, keeping the coefficient functions $\beta_k(x)$ unknown. We can restrict those coefficients to satisfy:

$$
\underset{y\to 0}{PWET_k}\{{}^{y}x\} = 0
$$

for all nonnegative integer *k*. If we only require that this is true up to some integer *n*, where $0 \le k < n$ then we get a rather small system of equations, say for $n = 2$:

$$\left\{\begin{array}{c} -1 + x \cdot x^{\beta_1(x)} \cdot x^{\beta_2(x)/2} = 0 \\ -\beta_1(x) + x \cdot x^{\beta_1(x)} \cdot x^{\beta_2(x)/2} \cdot \log(x) \cdot (\beta_1(x) - \beta_2(x)) = 0 \end{array}\right\}$$

which are nonlinear equations, and in general, are hard to solve, even with a computer. The reason why only two unknown terms were used in the equations above, is that the zeroth term, or $\beta_0(x) = {}^0x = 1$ is already known, and we only have two equations. Solving systems of equations works best when the number of unknowns is the same as the number of equations, so given two equations we should be able to solve for two unknowns. The solutions to equations obtained in this way for an extension of tetration, generally have very extreme values, and increasing the degree to $n = 3$, for example, will produce solutions for $\beta_k(x)$ that differ greatly. Producing this kind of system of equations for the super-logarithm, on the other hand, is more well-behaved.

Starting with definition (2) and definition (5) instead, we will use the series extension as the critical function of the piecewise extension of the super-logarithm, letting the coefficients $v_k(y)$ of the series remain unknown. We already know the zeroth coefficient: $v_0(x) = \mathrm{slog}_x(0) = -1$ from integer tetration, so we will be solving for the coefficients $v_{k+1}(x)$ where $0 \le k < n$ in the equations generated by letting the piecewise error transform of the super-logarithm equal zero. For $n = 2$:

$$\left\{\begin{array}{c} 1 - v_1 - \frac{1}{2}v_2 = 0 \\ v_1 - \log(x)v_1 - \log(x)v_2 = 0 \end{array}\right\}$$

For $n = 3$:

$$\left\{\begin{array}{c} 1 - v_1 - \frac{1}{2}v_2 - \frac{1}{6}v_3 = 0 \\ v_1 - \log(x)v_1 - \log(x)v_2 - \frac{1}{2}\log(x)v_3 = 0 \\ -\log(x)^2 v_1 + v_2 - 2\log(x)^2 v_2 - \frac{3}{2}\log(x)^2 v_3 = 0 \end{array}\right\}$$

where $v_k = v_k(x)$. This time, however, the equations are linear, so there are many more methods at our disposal for determining if the equations are solvable, and finding the solution. One such way is finding the determinant of the matrix associated with the equations. Before we can turn the system of equations into a matrix we must put only the unknowns on the left, and constants on the right. As you can see in the systems above the only constant is the 1 in the first equation in each system. For the equations found above for $n = 2$, we can group them like this:

$$\left\{\begin{array}{rrr} -v_1 & -\frac{1}{2}v_2 = & -1 \\ (1-\log(x))v_1 & -\log(x)v_2 = & 0 \end{array}\right\}$$

Also, in the interest of readability all equations will be divided by a power of $\log(x)$, because as you can see above, each equation has an increasing power of $\log(x)$ in it. Also, because the majority of the coefficients are negative we can also reverse the sign of the equations. Reversing the sign, and dividing by a power of $\log(x)$ will make:

$$\left\{\begin{array}{rrr} v_1 & +\frac{1}{2}v_2 = & 1 \\ \left(1-\frac{1}{\log(x)}\right)v_1 & +v_2 = & 0 \end{array}\right\}$$

but, before we can represent the above set of equations as a matrix we must define a basis. As we stated earlier, the unknowns we are solving for are related to the coefficients of the series in definition (5). These unknowns are also the derivatives of the super-logarithm at $z = 0$, but as a basis, they are not merely numbers or vectors, they are functions of x. Using the sequence notation $\langle\cdot\rangle_{k=i}^{n}$, the basis we will be using is:

$$\mathbf{v} = \langle v_k\rangle_{k=1}^{n} \text{ where } v_k = v_k(x) = \boldsymbol{D}_z^k[s(x,z)_n]_{z=0}$$

Using this basis, the above system of equations has the matrix equation for $n = 2$:

$$\left\langle\frac{-\underset{z\to 0}{PWET}_k\{\mathrm{slog}_x(z)_2\}}{\log(x)^k}\right\rangle_{k=0}^{1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 1-\frac{1}{\log(x)} & 1 \end{bmatrix}\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $n = 3$:

$$\left\langle \frac{\underset{z\to 0}{-PWET_k}\{\mathrm{slog}_x(z)_3\}}{\log(x)^k} \right\rangle_{k=0}^{2} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{6} \\ 1-\frac{1}{\log(x)} & 1 & \frac{1}{2} \\ 1 & 2-\frac{1}{\log(x)^2} & \frac{3}{2} \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $n = 4$:

$$\left\langle \frac{\underset{z\to 0}{-PWET_k}\{\mathrm{slog}_x(z)_4\}}{\log(x)^k} \right\rangle_{k=0}^{3} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 1-\frac{1}{\log(x)} & 1 & \frac{1}{2} & \frac{1}{6} \\ 1 & 2-\frac{1}{\log(x)^2} & \frac{3}{2} & \frac{2}{3} \\ 1 & 4 & \frac{9}{2}-\frac{1}{\log(x)^3} & \frac{8}{3} \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For $n = 5$:

$$\left\langle \frac{\underset{z\to 0}{-PWET_k}\{\mathrm{slog}_x(z)_5\}}{\log(x)^k} \right\rangle_{k=0}^{4} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ 1-\frac{1}{\log(x)} & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 1 & 2-\frac{1}{\log(x)^2} & \frac{3}{2} & \frac{2}{3} & \frac{5}{24} \\ 1 & 4 & \frac{9}{2}-\frac{1}{\log(x)^3} & \frac{8}{3} & \frac{25}{24} \\ 1 & 8 & \frac{27}{2} & \frac{32}{3}-\frac{1}{\log(x)^4} & \frac{125}{24} \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\mathrm{slog}_x(z)_n$ is the piecewise extension of the super-logarithm with an analytic extension as its critical function whose coefficients are obtained from the system of n equations in n unknowns, generated by letting the piecewise error transform equal zero. The equation matrices above can be computed without finding the derivatives of the super-logarithm. An alternate way of generating the above matrix is:

$$\left\langle \left\langle \frac{m^k}{m!} - \delta_{mk}\log(x)^{-k} \right\rangle_{m=1}^{n} \right\rangle_{k=0}^{n-1}$$

where δ_{jk} is the Kronecker delta (1 if $j = k$, 0 otherwise), usually used with tensors.

To indicate that the critical function used by the general piecewise extension is found at a certain value of *n*, the notation: $s(x,z)_n$ will be used. Now that we can solve for an extension of the super-logarithm, what do the solutions look like? First, the solution when $n = 1$ is actually the same as extension (2):

$$
\begin{aligned}
&v_1(x)_1 = 1 \\
&s(x,z)_1 = -1\cdot z^0 + v_1(x)_1 \cdot z^1 \\
&s(x,z)_1 = -1 + z
\end{aligned}
$$

For $n = 2$:

$$s(x,z)_2 = -1 + \frac{2\log(x)}{1+\log(x)}z - \frac{1-\log(x)}{1+\log(x)}z^2$$

For $n = 3$:

$$s(x,z)_3 = -1 + \frac{6[\log(x)+\log(x)^3]z + 3[3\log(x)^2 - 2\log(x)^3]z^2 + 2[1-\log(x)-2\log(x)^2+\log(x)^3]z^3}{2+4\log(x)+5\log(x)^2+2\log(x)^3}$$

Assuming that the system of equations are always linear, we can find any degree solution we want, given enough time. What exactly have we found, though? Solving for the super-logarithm using the piecewise error transform ensures that the function found will be differentiable up to a point.

Lemma 1. If $\underset{x\to c}{PWET_k}\{f(x)\} = 0$ for $0 \le k < n$, then $f(x)$ is $\boldsymbol{C}^{n-1}$.

Even if we do find that the functions we get from the solutions to the system of equations are $\boldsymbol{C}^n$, we still do not know if the solution is unique. To find out whether the solution is unique we can use the determinant of the matrix that expresses the system of equations. When the determinant is zero, then the solution is not unique, when the determinant is not zero, then there must be one and only one solution. Here are the determinants of the simplified matrices obtained from the systems of equations generated by letting the piecewise error transform of the super-logarithm equal zero, for $n = 2$:

$$\det\left\langle \frac{-\underset{z\to 0}{PWET_k}\{\mathrm{slog}_x(z)_2\}}{\log(x)^k} \right\rangle_{k=0}^{1} = \frac{1}{2} + \frac{1}{2\log(x)}$$

For $n = 3$:

$$\det\left\langle \frac{-\underset{z\to 0}{PWET_k}\{\mathrm{slog}_x(z)_3\}}{\log(x)^k} \right\rangle_{k=0}^{2} = \frac{1}{6} + \frac{5}{12\log(x)} + \frac{1}{3\log(x)^2} + \frac{1}{6\log(x)^3}$$

In order to use these expressions to find when the systems of solutions are solvable and have unique solutions, we need to find when these expressions are equal to zero. When $x > 1$, $\log(x)$ is positive, and since all the coefficients in the determinants are positive, the only way the whole determinant will be zero is if $\log(x)$ is negative. Since this only happens when $0 < x < 1$, the determinant is nonzero for $x > 1$. Some of the roots of these determinants for different n are given here to illustrate:

n	***x***
2	0.367879
3	0.190653
4	0.126582, 0.494301
5	0.099918, 0.323049

and as you can see from the table, all roots seem to be between zero and one.

Lemma 2. $\det\left(\left\langle \underset{z\to 0}{PWET_k}\{\mathrm{slog}_x(z)_n\} \right\rangle_{k=0}^{n-1}\right) = 0$ implies $0 < x < 1$.

Going back to the piecewise error transform, implicitly declared in letting its application on the super-logarithm equal zero is the relationship:

$$v_k(x) = \boldsymbol{D}_z^k[s(x,z)]_{z=0} = \boldsymbol{D}_z^k[s(x,x^z) - 1]_{z=0}$$

which can be used to simplify the series expansion of the critical function used by the piecewise definition. We can now define an extension of the super-logarithm as:

Definition 7. *The analytic piecewise extension of the super-logarithm*

$$s(x,z)_n = -1 + \sum_{k=1}^{n} \frac{z^k}{k!} v_k(x) \qquad \text{if } x > 1,\ 0 < z \le 1$$

$$\text{where} \quad v_k(x) = \boldsymbol{D}_z^k [s(x, x^z)_n - 1]_{z=0}$$

$$\text{or} \quad v_k(x) = \left[\left\langle \left\langle \frac{m^k}{m!} - \delta_{mk} \log(x)^{-k} \right\rangle_{m=1}^{n} \right\rangle_{k=0}^{n-1} \right]^{-1} \left\langle \delta_{m1} \right\rangle_{m=1}^{n}$$

$$\operatorname{slog}_x(z)_n = s(x, x^z)_n - 1 \qquad \text{if } x > 1,\ z \le 0$$

$$\operatorname{slog}_x(z)_n = s(x, z)_n \qquad \text{if } x > 1,\ 0 < z \le 1$$

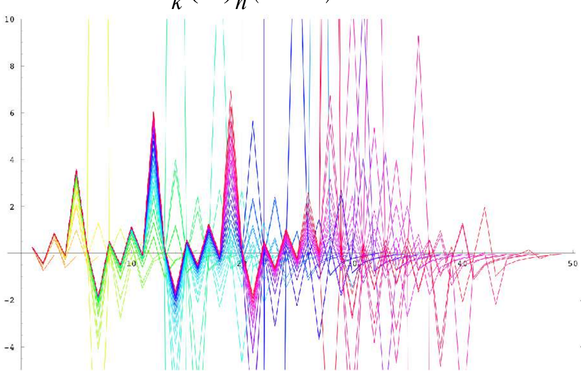
$$\operatorname{slog}_x(z)_n = s(x, \log_x^m(z))_n + m \qquad \text{if } x > 1,\ ({}^{m-1}x) < z \le ({}^{m}x),\ m > 0$$

$$\operatorname{slog}_x(z) = \lim_{n \to \infty} \operatorname{slog}_x(z)_n \qquad \text{if } x > 1$$

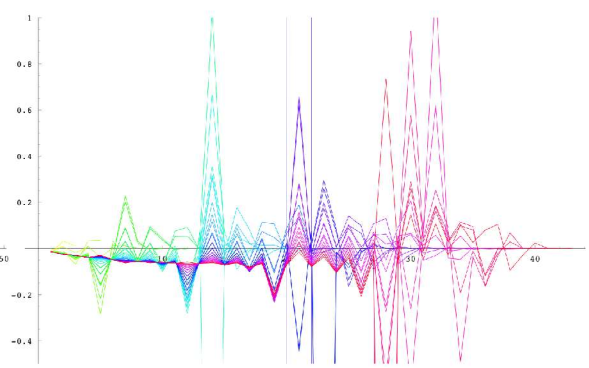
where the superscript (–1) indicates the inverse matrix.

We already know that the super-logarithm defined in this manner will satisfy property (2), because it is a piecewise extension. Although we can soon find if the super-logarithm satisfies property (3), it still is not obvious that the series converges, or that the final limit exists. To show this, we need to take a closer look at $v_k(x)$. Applying the ratio test for when $x = \boldsymbol{e}$, we get a rather chaotic graph. Looking closely at this graph, there is a pattern that repeats every seven terms, so we find that the graph of the ratios of the terms is much smoother when the ratio is for those terms that are seven terms apart:

graph of $\dfrac{v_{k+1}(\boldsymbol{e})_n}{v_k(\boldsymbol{e})_n (k+1)}$ in k from 1 to n



graph of $\dfrac{v_{k+7}(\boldsymbol{e})_n k!}{v_k(\boldsymbol{e})_n (k+7)!}$ in k from 1 to n



where the second graph seems to fall on a curve for low k. This curve can be expressed algebraically by the approximate recurrence relations:

$$v_{k+7}(\boldsymbol{e}) \approx -v_k(\boldsymbol{e})\boldsymbol{e}^{2\pi} k^{2.95} k^{\log\sqrt{k}}$$

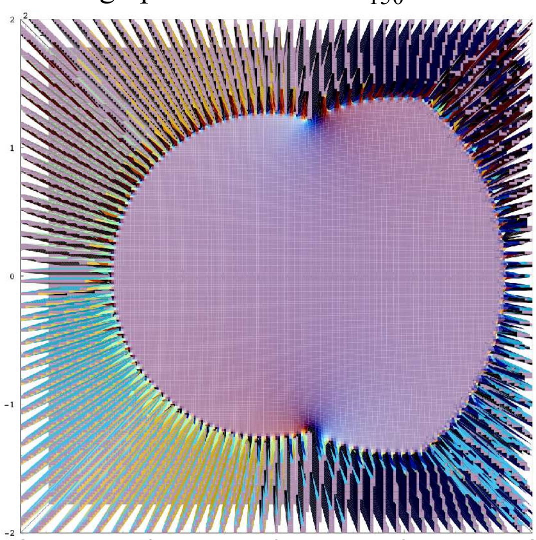
or: $$v_{k+7}(\boldsymbol{e}) \approx -v_k(\boldsymbol{e}) k^{7(7/k)^{1/k}}$$

where the magnitudes are roughly:

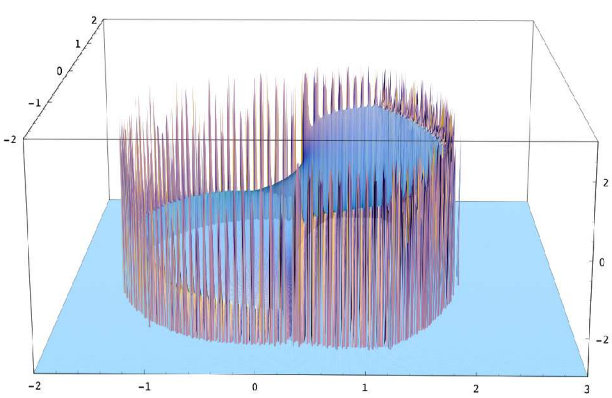
$$\left|v_k(\boldsymbol{e})\right| \approx k^{(k^2/2950 + k/\sqrt{2} - 3)}$$

which helps explain the signs of the terms. These do not help us determine the convergence of the series, because in the limit as $k \to \infty$ they become infinite. For now, we can only see whether the approximations of the super-logarithm constructed as in definition (7), converge to each other. In order to do this we will look only at the critical function $s(\boldsymbol{e}, z)_{150}$. The definition of this function is not limited to real numbers, it is only the piecewise-defined $\mathrm{slog}_x(z)_n$ that is limited to real numbers. Outside a certain region in the complex plane, complex z will give extreme values that do not correspond to previous approximations, so we can identify an approximate radius of convergence by looking at some 3D plots of the *real part* of the critical function $s(\boldsymbol{e}, z)_{150}$:

graph of $\Re[s(\boldsymbol{e}, z)_{150}]$



graph of $\Re[s(\boldsymbol{e}, z)_{150}] = y$ if $y < 2$



as you can see, although we haven't found the radius of convergence algebraically, there does seem to be a radius of convergence, and it covers the required domain of the critical function: $0 \le z \le 1$. These graphs and the tables of numerical data in appendix C are the best proof of convergence that the author has found. Hopefully a more exact method will be found, but until then we will assume it converges. There is no indication that it would not converge, but that is not enough to prove it does.

Assuming $\lim_{n \to \infty} s(x, z)_n$ converges for all $0 \le z \le 1$, the following shows the uniqueness and properties of the super-logarithm in definition (7):

Rough Proof

The critical functions $s(x, z)_n$ converge for finite *n*, and $\lim_{n \to \infty} s(x, z)_n$ seems to converge within the approximate radius of convergence shown above. There is no question if the critical functions are $\boldsymbol{C}^{\infty}$. The piecewise-defined functions $\mathrm{slog}_x(z)_n$ are $\boldsymbol{C}^{n-1}$, by lemma (1), so in the limit $\mathrm{slog}_x(z)_\infty$ is $\boldsymbol{C}^{\infty}$. This proves that $\mathrm{slog}_x(z)_\infty$ satisfies property (3), and since the piecewise-defined functions $\mathrm{slog}_x(z)_n$ are defined using property (2), they must satisfy it. Since the determinant being zero implies $0 < x < 1$, by lemma (2), the logical complement would be that the determinant is nonzero if $x > 1$. The determinant being nonzero then implies there is one and only one solution to the system of equations the matrix represents. So if $x > 1$, $\mathrm{slog}_x(z)_\infty$ exists and is unique. Therefore, given an $x > 1$, $\mathrm{slog}_x(z) = \mathrm{slog}_x(z)_\infty$ exists, is unique, and satisfies properties (2) and (3). This proves that definition (7) is *the* analytic piecewise extension of the super-logarithm to all real number *z*. ∎

Generalization

Solving for the super-logarithm has produced “nice” C^n approximations. We can generalize this by going back to Abel's equation for linearizing a function:

$$\begin{aligned}
&\phi(f(z)) = \phi(z) + 1\\
&\phi(z) = \phi(f(z)) - 1\\
&\boldsymbol{D}^k[\phi(z)] = \boldsymbol{D}^k[\phi(f(z)) - 1]\\
&\boldsymbol{D}^k[\phi(z)]_{z=0} = \boldsymbol{D}^k[\phi(f(z)) - 1]_{z=0}
\end{aligned}$$

$$\begin{aligned}
&\phi(z)_n = \sum_{k=0}^{n} \frac{z^k}{k!} \boldsymbol{D}^k[\phi(f(z)) - 1]_{z=0}\\
&\phi(z) = \lim_{n\to\infty} \phi(z)_n
\end{aligned}$$

defining the linearizing function by the property we desire of it, and solving for approximations of it for some $0 \le k < n$. It is possible this could work for any analytic $f(z)$, but it is likely that additional conditions apply. If the equations have a solution, and the solutions converge, then any iterate of the function can be expressed as:

$$\begin{aligned}
&\phi(f(z)) = \phi(z) + 1\\
&\phi(f^n(z)) = \phi(z) + n\\
&f^n(z) = \phi^{-1}(\phi(z)+n)
\end{aligned}$$

This method is not restricted to continuously iterating a function over the real numbers, the series expansion should work within some complex radius of convergence.

Conclusion

We now have a unique analytic piecewise extension of the super-logarithm. Using the super-logarithm, we can define tetration as its inverse. Also, because the super-logarithm is the linearizing function of exponentials, we can use it to define continuously iterated exponentiation as well. Remember that the first definition of the super-logarithm requires that ${}^{\mathrm{slog}_x(a)}x = a = \exp_x^{\mathrm{slog}_x(a)}(1)$ hold in general. This allows the definition of continuously iterated exponentiation as follows:

Definition 8. *Tetration*

$$
\begin{aligned}
{}^{y}x &= \mathrm{slog}_x^{-1}(y) && \text{if } x > 1,\ y > -2 \\
{}^{y}x &= \log_x({}^{y+1}x) && \text{if } x > 1,\ y \le -2
\end{aligned}
$$

Definition 9. *Continuously iterated exponentiation*

$$
\exp_x^{y}(a) = {}^{y+\mathrm{slog}_x(a)}x \quad \text{if } x > 1
$$

In this paper, a unique analytic piecewise extension has been presented for the super-logarithm, tetration, and continuously iterated exponentiation. In appendix A, the method used for solving the systems of equations to produce the approximations of these extensions is given. Now that the numerical values of the super-logarithm and tetration can be found to any precision, it is only a matter of time before a closed-form definition of tetration and related functions is found.

Appendix A — Code

Note from the Author

There are two algorithms for finding super-log. One is to compute the derivatives, and the other it so compute the matrix. Both methods require solving a system of equations, whether in matrix form or not. The matrix method takes about the same time as computing the derivatives symbolically, so their is not much performance gain. Since it takes so much time to compute the solutions for higher approximations, I made a separate function to compute them rather than include them in the functions themselves. This allows you to wait for the solutions once, and use them over and over rather quickly. Since the base is part of the coefficients, it must be given before solving. I have not made an implementation of the super-root, because it involves solving for any base symbolically, which in general takes much longer than finding it for a numerical base. Of all the bases, ***e*** requires the least time to prepare for, because all the equations being solved become rational. Some sample commands are given after the implementations, some of which demonstrate the numerical derivative functions included to reproduce some of the graphs given in this text. You might have to mess with h for higher derivatives, because the default $h = 0.0001$ and this produces almost random 4th and 5th derivatives. To fix this, let $h = 0.001$ or $h = 0.01$ and they become more smooth.

In Pseudo-code

let *s*(*x*, *z*, *n*) = -1 + sum[*k* = 1..*n*] *z*^*k*/*k*! *v*_*k*(*x*)

solve for *v*_(*k*+1)(*x*) where *k* = 0..(*n*-1) in:

  (***D***_*z*^*k*[*s*(*x*, *z*, *n*)]@*z*=0 – ***D***_*z*^*k*[*s*(*x*, *x*^*z*, *n*)–1]@*z*=0) = 0

where *x* > 1, let slog(*x*, *z*, *n*) = one of:

s(*x*, *x*^*z*, *n*) – 1	if *z* <= 0
s(*x*, *z*, *n*)	if 0 < *z* <= 1
s(*x*, log_*x*(*z*), *n*) + 1	if 1 < *z* <= *x*
s(*x*, log_*x*(log_*x*(*z*)), *n*) + 2	if *x* < *z* <= *x*^*x*
...	

let tetrate(*x*, *y*, *n*) = *z* where *y* = slog(*x*, *z*, *n*).

In Maple

```
## Usage:
##      env := superlog_prepare(n, x):
##      superlog(env, z);   -- gives n-th approx. of slog_x(z)
##      tetrate(env, y);    -- gives n-th approx. of x^^y
## Copyright 2005 Andrew Robbins

with(linalg):

superlog_prepare := proc(n::integer, x)
    [x, linsolve(matrix([seq([seq(
        k^j/k! - `if`(j = k, 1, 0)*log(x)^(-j),
        k = 1..n)], j = 0..(n - 1))]),
        [seq(`if`(k = 1, 1, 0), k = 1..n)])];
end proc;

superlog := proc(v, z) local slog_crit;
    if not (z::numeric) then return 'procname'(args); end if;
    slog_crit := proc(zc) -1 + sum(v[2][k]*zc^k/k!,
        k = 1..(vectdim(v[2]))); end proc;
    piecewise(z = -infinity, -2,
        z < 0, slog_crit(v[1]^z) - 1, z = 0, -1,
        0 < z and z < 1, slog_crit(z), z = 1, 0,
        z > 1, (proc() local a, i; a := z;
            for i from 0 while (evalf(a) > 1) do a := log[v[1]](a); end do;
            slog_crit(evalf(a)) + i; end proc)());
end proc;

tetrate := proc(v, y) local tet_crit;
    if not (y::numeric) then return 'procname'(args); end if;
    tet_crit := proc(yc) local slog_crit;
        slog_crit := proc(zc) -1 + sum(v[2][k]*zc^k/k!,
            k = 1..(vectdim(v[2]))); end proc;
        select((proc(a) evalb(Im(a) = 0 and 0 <= Re(a) and Re(a) <= 1)
            end proc), [solve(evalf(slog_crit(z)) = yc, z)])[1];
    end proc;
    piecewise(y = -2, -infinity,
        -2 < y and y < -1, log[v[1]](tet_crit(y+1)), y = -1, 0,
        -1 < y and y < 0, tet_crit(y), y = 0, 1,
        y > 0, (proc () local a, i; a := tet_crit(y - ceil(y));
            for i from 1 to ceil(y) do a := v[1]^a; end do;
            evalf(a); end proc)());
end proc;
```

```
## Gives the k-th numerical derivative of f(x) at x=c:

    ndiff := proc(f, x, c, k, h)
       if (k = 0) then subs(x = c, f);
       else (ndiff(f, x, c+h, k-1, h) - ndiff(f, x, c, k-1, h))/h;
       end if;
    end proc;

#########################################
## A few commands to try for starters: ##
#########################################

## Prepares the 10th approx. of slog_e(z):
##  the ':' is used here to suppress display

    env := superlog_prepare(10, exp(1)):

## Shows e^^0.5, and plots of e^^y:

    tetrate(env, 0.5);
    plot(tetrate(env, y), y = -2..2, view = -5..15);
    plot([tetrate(env, y), ndiff(tetrate(env, t), t, y, 1, 0.0001)],
        y = -2..2, view = -5..10);

## Shows slog_e(1.5), and plots of slog_e(z):

    superlog(env, 1.5);
    plot(superlog(env, z), z = -5..15, view = -2..2);
    plot([superlog(env, z), ndiff(superlog(env, t), t, z, 1, 0.0001)],
        z = -5..10, view = -2..2);

## Shows the half-exponential function:
##  f(x) such that f(f(x)) = exp(x)

    half_exp := proc(x) tetrate(env, superlog(env, x) + 1/2); end proc;
    plot([x, half_exp(x), half_exp(half_exp(x))],
        x = -2..4, view = -1..5);

## Plots of slog at different bases:

    plot([superlog(superlog_prepare(5, 2), z),
        superlog(superlog_prepare(5, 10), z)],
        z = -4..8, view = -2..3);
```

In Mathematica

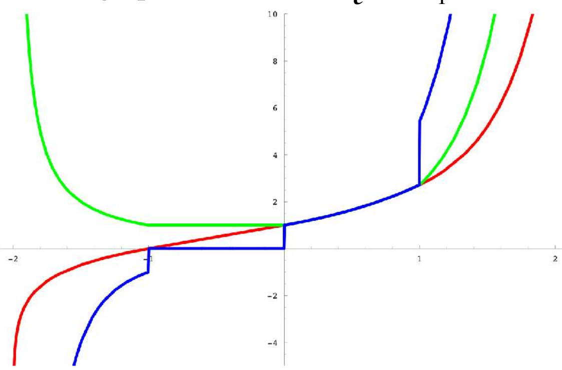
```
(*
** Usage:
**      env = SuperLogPrepare[n, x];
**      SuperLog[env, z]    -- gives n-th approx. of slog_x(z)
**      Tetrate[env, y]     -- gives n-th approx. of x^^y
** Copyright 2005 Andrew Robbins
*)

SuperLogPrepare[n_Integer, x_] := {x, LinearSolve[Table[
    k^j/k! - If[j == k, Log[x]^-k, 0], {j, 0, n - 1},
    {k, 1, n}], Table[If[k == 1, 1, 0], {k, 1, n}]]}

SuperLog[v_, z_?NumericQ] := Block[{(*SlogCrit*)},
    SlogCrit[zc_] := -1 + Sum[v[[2, k]]*zc^k/k!, {k, 1, Length[v[[2]]]}];
    Which[z ≤ 0, SlogCrit[v[[1]]^z] - 1, 0 < z ≤ 1, SlogCrit[z], z > 1,
    Block[{i=-1}, SlogCrit[NestWhile[Log[v[[1]], #]&, z, (i++;#>1)&]]+i]]]

Tetrate[v_, y_?NumericQ] := Block[{(*SlogCrit, TetCrit*)},
    SlogCrit[zc_] := -1 + Sum[v[[2, k]]*zc^k/k!, {k, 1, Length[v[[2]]]}];
    TetCrit[yc_] := FindRoot[SlogCrit[z] == yc, {z, 1}][[1, 2]]; If[y > -1,
    Nest[Power[v[[1]], #]&, TetCrit[y - Ceiling[y]], Ceiling[y]],
    Nest[Log[v[[1]], #]&, TetCrit[y - Ceiling[y]], -Ceiling[y]]]]

(* Gives the k-th numerical derivative of f(x) at x=c: *)
ND[f_, x_, c_] := ND[f, x, c, 1]
ND[f_, x_, c_, k_] := ND[f, x, c, k, 0.0001]
ND[f_, x_, c_, 0, h_] := (f /. x -> c)
ND[f_, x_, c_, k_, h_] /; k != 0 := (
    ND[f, x, c+h, k-1, h] - ND[f, x, c, k-1, h])/h
```

```
(***************************************)
(* A few commands to try for starters: *)
(***************************************)

(* Prepares the 10th approx. of slog_e(z): *)
(*  the '=' is used here for immediate execution *)
(*  the ';' is used here to suppress display *)

    env = SuperLogPrepare[10, E];

(* Shows e^^0.5, and plots of e^^y: *)

    Tetrate[env, 0.5]
    Plot[Tetrate[env, y], {y, -2, 2}, PlotRange -> {-5, 15}]
    Plot[{Tetrate[env, y], ND[Tetrate[env, t], t, y]},
        {y, -2, 2}, PlotRange -> {-5, 10},
        PlotStyle -> {Hue[0], Hue[2/3]}]

(* Shows slog_e(1.5), and plots of slog_e(z): *)

    SuperLog[env, 1.5]
    Plot[SuperLog[env, z], {z, -5, 15}, PlotRange -> {-2, 2}]
    Plot[{SuperLog[env, z], ND[SuperLog[env, t], t, z]},
        {z, -5, 10}, PlotRange -> {-2, 2},
        PlotStyle -> {Hue[0], Hue[2/3]}]

(* Shows the half-exponential function: *)
(*  f(x) such that f(f(x)) = exp(x) *)

    HalfExp[x_] := Tetrate[env, SuperLog[env, x] + 1/2]
    Plot[{x, HalfExp[x], HalfExp[HalfExp[x]]},
        {x, -2, 4}, PlotRange -> {-1, 5}]

(* Plots 5th approx. of slog at different bases: *)

    Plot[{SuperLog[SuperLogPrepare[5, 2], z],
        SuperLog[SuperLogPrepare[5, 10], z]},
        {z, -4, 8}, PlotRange -> {-2, 3},
        PlotStyle -> {Hue[0], Hue[2/3]}]
```

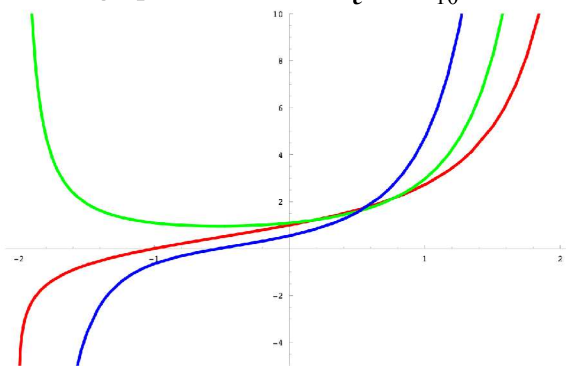
Appendix B — Graphs

Here are graphs of tetration and the super-logarithm, to illustrate the difference between the linear critical extensions, and the new definitions. The red line is the function itself, the green line is the first derivative, and the blue line is the second derivative:

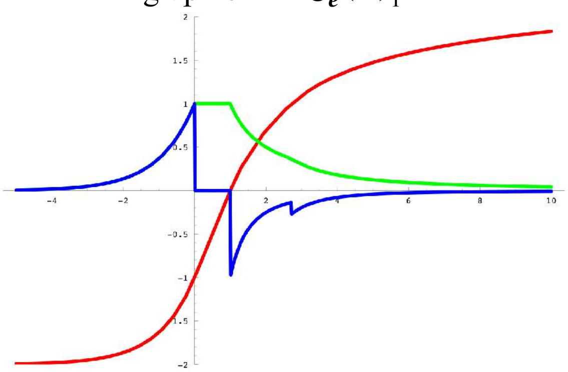
graph of ${}^{y}\boldsymbol{e} \approx \text{slog}_{\boldsymbol{e}}^{-1}(y)_1$



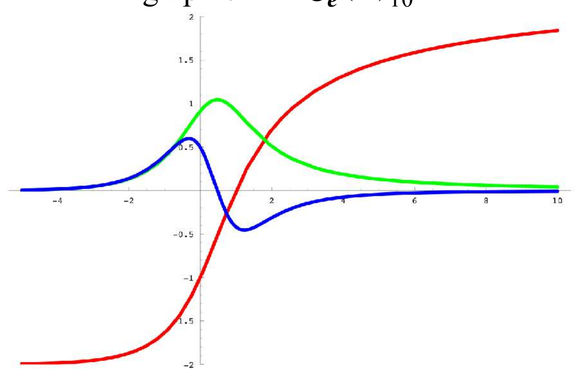
graph of ${}^{y}\boldsymbol{e} \approx \text{slog}_{\boldsymbol{e}}^{-1}(y)_{10}$



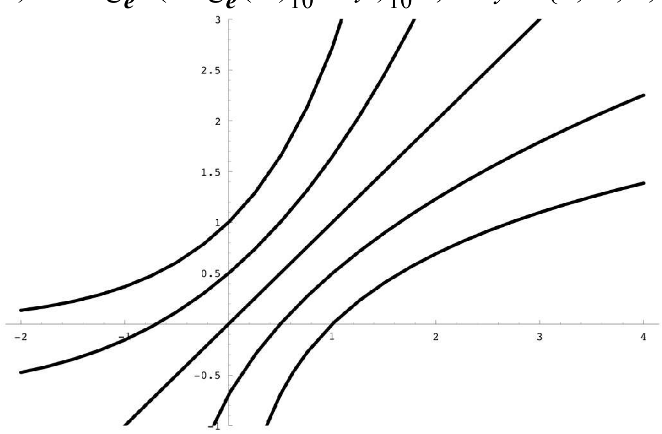
graph of $\text{slog}_{\boldsymbol{e}}(z)_1$



graph of $\text{slog}_{\boldsymbol{e}}(z)_{10}$



A graph of $\exp^{y}(x) \approx \text{slog}_{\boldsymbol{e}}^{-1}(\text{slog}_{\boldsymbol{e}}(x)_{10} + y)_{10}$, for $y = \{1, ½, 0, -½, -1\}$:



Appendix C — Numerical Data

$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{40}$	$= 0.915944781172534$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{40}$	$= 0.498696444588079$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{50}$	$= 0.915945263266249$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{50}$	$= 0.498704465665853$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{60}$	$= 0.915945536274640$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{60}$	$= 0.498707053886320$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{70}$	$= 0.915945731075678$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{70}$	$= 0.498708123849263$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{80}$	$= 0.915945846988776$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{80}$	$= 0.498708709807559$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{90}$	$= 0.915945908914126$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{90}$	$= 0.498708986251720$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{100}$	$= 0.915945951095597$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{100}$	$= 0.498709096804952$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{110}$	$= 0.915945982447644$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{110}$	$= 0.498709165188107$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{120}$	$= 0.915946001992852$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{120}$	$= 0.498709210043943$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{130}$	$= 0.915946014757482$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{130}$	$= 0.498709225011490$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{140}$	$= 0.915946025105883$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{140}$	$= 0.498709229740035$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)_{150}$	$= 0.915946032676321$	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)_{150}$	$= 0.498709235488693$
$\boldsymbol{D}^1 \operatorname{slog}_{\boldsymbol{e}}(0)$	≈ 0.91594603	$\boldsymbol{D}^2 \operatorname{slog}_{\boldsymbol{e}}(0)$	≈ 0.49870923

$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{40}$	$= -0.66276507305193$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{40}$	$= -2.25390534517765$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{50}$	$= -0.66276873212276$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{50}$	$= -2.25423131435349$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{60}$	$= -0.66277398975381$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{60}$	$= -2.25434721926022$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{70}$	$= -0.66277900884800$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{70}$	$= -2.25440263020639$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{80}$	$= -0.66278207800690$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{80}$	$= -2.25443382098240$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{90}$	$= -0.66278377739784$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{90}$	$= -2.25444919899063$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{100}$	$= -0.66278506194145$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{100}$	$= -2.25445694299939$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{110}$	$= -0.66278603921500$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{110}$	$= -2.25446221464863$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{120}$	$= -0.66278664482280$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{120}$	$= -2.25446557924414$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{130}$	$= -0.66278706373557$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{130}$	$= -2.25446727337134$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{140}$	$= -0.66278741544855$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{140}$	$= -2.25446838669198$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)_{150}$	$= -0.66278766900692$	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)_{150}$	$= -2.25446928157285$
$\boldsymbol{D}^3 \operatorname{slog}_{\boldsymbol{e}}(0)$	≈ -0.662787	$\boldsymbol{D}^4 \operatorname{slog}_{\boldsymbol{e}}(0)$	≈ -2.25446

$$\begin{aligned}
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{40} &&= 1.20123676054844 &\quad &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{40} &&= 25.8189589561658\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{50} &&= 1.20028806575165 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{50} &&= 25.8329970026009\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{60} &&= 1.20013405434286 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{60} &&= 25.8387294460917\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{70} &&= 1.20017665549235 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{70} &&= 25.8419419969967\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{80} &&= 1.20021193383909 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{80} &&= 25.8437963173890\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{90} &&= 1.20023796411121 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{90} &&= 25.8447456907652\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{100} &&= 1.20027097420290 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{100} &&= 25.8453047049794\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{110} &&= 1.20029821853748 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{110} &&= 25.8457046662579\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{120} &&= 1.20031476521818 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{120} &&= 25.8459565204653\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{130} &&= 1.20032838768356 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{130} &&= 25.8461048506513\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{140} &&= 1.20034088728237 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{140} &&= 25.8462167550629\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0)_{150} &&= 1.20034958138708 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0)_{150} &&= 25.8463012017336\\
&\boldsymbol{D}^5\operatorname{slog}_{\boldsymbol{e}}(0) &&\approx 1.20034 & &\boldsymbol{D}^6\operatorname{slog}_{\boldsymbol{e}}(0) &&\approx 25.846
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{40} &&= 32.9490742321220 &\quad &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{40} &&= -495.05095328987\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{50} &&= 33.0744463270470 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{50} &&= -495.47688098167\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{60} &&= 33.1106415509182 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{60} &&= -495.72547010186\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{70} &&= 33.1226292184246 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{70} &&= -495.90651130833\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{80} &&= 33.1288636927751 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{80} &&= -496.01445390533\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{90} &&= 33.1315430903712 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{90} &&= -496.07230375701\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{100} &&= 33.1319812761822 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{100} &&= -496.11212255466\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{110} &&= 33.1320611530065 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{110} &&= -496.14179306528\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{120} &&= 33.1321507609728 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{120} &&= -496.16027838058\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{130} &&= 33.1319536106881 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{130} &&= -496.17242819312\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{140} &&= 33.1316616124526 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{140} &&= -496.18231831220\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0)_{150} &&= 33.1314888709522 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0)_{150} &&= -496.18954127850\\
&\boldsymbol{D}^7\operatorname{slog}_{\boldsymbol{e}}(0) &&\approx 33.131 & &\boldsymbol{D}^8\operatorname{slog}_{\boldsymbol{e}}(0) &&\approx -496.18
\end{aligned}$$

$\mathrm{slog}_2^{-1}(0.5)_2$	$= 1.458961693832438$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(0.5)_{10}$	$= 1.6464556716360208$
$\mathrm{slog}_2^{-1}(0.5)_4$	$= 1.458655904880133$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(0.5)_{20}$	$= 1.6463676325953218$
$\mathrm{slog}_2^{-1}(0.5)_6$	$= 1.458692450371729$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(0.5)_{30}$	$= 1.6463577769479243$
$\mathrm{slog}_2^{-1}(0.5)_8$	$= 1.458741984415978$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(0.5)_{40}$	$= 1.6463553806741427$
$\mathrm{slog}_2^{-1}(0.5)_{10}$	$= 1.458768532260076$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(0.5)_{50}$	$= 1.6463546427466649$
${}^{0.5}2$	≈ 1.4587	${}^{0.5}\boldsymbol{e}$	≈ 1.64635

$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{10}$	$= 2078.198719173609$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{10}$	$= 37105406757.56952$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{20}$	$= 2076.129166296636$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{20}$	$= 37155268624.63599$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{30}$	$= 2075.998583292668$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{30}$	$= 37152290690.85273$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{40}$	$= 2075.975284589419$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{40}$	$= 37150849430.35024$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{50}$	$= 2075.968983446195$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{50}$	$= 37150331380.03964$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{60}$	$= 2075.967658498696$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{60}$	$= 37150112554.57623$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{70}$	$= 2075.967604923759$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{70}$	$= 37149986051.50005$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{80}$	$= 2075.967631271361$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{80}$	$= 37149912712.49439$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{90}$	$= 2075.967687365403$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{90}$	$= 37149874928.20818$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{100}$	$= 2075.967814831020$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{100}$	$= 37149852157.21806$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{110}$	$= 2075.967925399709$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{110}$	$= 37149835758.34629$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{120}$	$= 2075.967991775568$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{120}$	$= 37149825450.22689$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{130}$	$= 2075.968051571147$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{130}$	$= 37149819264.78736$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{140}$	$= 2075.968108549399$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{140}$	$= 37149814532.74459$
$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{e})_{150}$	$= 2075.968147604069$	$\mathrm{slog}_{\boldsymbol{e}}^{-1}(\pi)_{150}$	$= 37149810983.64210$
${}^{\boldsymbol{e}}\boldsymbol{e}$	≈ 2075.9681	${}^{\pi}\boldsymbol{e}$	$\approx 3.714981 \times 10^{10}$

Appendix D — Identities

$$^{-1}x = 0 \quad ^{0}x = 1 \quad ^{1}x = x \quad ^{2}x = x^{x}$$

$$^{y}x = \exp_x^y(1)$$
$$\exp_x^y(a) = {}^{y+\mathrm{slog}_x(a)}x$$

$$\mathrm{srt}_2(x) = 1/{}^{\infty}(x^{-1}) = \frac{\log(x)}{W(\log(x))}$$
$$\mathrm{srt}_\infty(x) = 1/{}^{2}(x^{-1}) = \sqrt[x]{x}$$

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